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Bose–Einstein condensation in an Einstein universe

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Abstract. The condensation of a non-relativistic ideal Bose gas in an Einstein universe is investigated. Explicit expressions for the condensate fraction N_0/N and the specific heat are obtained by using the Poisson summation formula to express the summations as integrations plus corrections. It is shown that the finiteness of the system smoothes out the cusp-like singularity of the infinite system. A rigorous asymptotic analysis of the critical temperature and the specific heat maximum are given, and the relation with the scaling theory of finite size effects is briefly discussed.

1. Introduction

In an infinite space a system of non-interacting bosons shows a critical thermal behaviour at very low temperatures. It is known that when the temperature is lowered sufficiently (about 3 K for ^{4}He) the particles seem to start accumulating in the lowest energy level. The specific heat is known to exhibit a discontinuity at this temperature. This phenomenon is called 'Bose–Einstein condensation' (full accounts are given in standard texts, e.g. London 1954).

In finite systems, however, the geometry of the system is known to have a distinct role in determining the thermal behaviour of bosons. Several authors have investigated this role played by the geometry, specifically in thin films. Early studies (Osborne 1949, Ziman 1953) have shown that a finite Bose–Einstein system exhibits a rapid accumulation of particles into the ground state at a more or less well defined temperature which was shown to be a function of the dimensions of the system (Ziman 1953). These studies and those that followed have served to elucidate the role played by the geometry of the system in determining the physical properties, especially near the condensation temperature T_0. Goble and Trainor (1966, 1967, 1968) studied the specific heat of a thin film of thickness D in a region near T_0. They found that the discontinuity of the bulk system is smoothed out and that the specific heat is a continuous function of the temperature being maximum at $T_0(D)$. Their study also showed that the specific heat maximum, $C_0(D)$, is itself a monotonic function of the thickness D of the thin film, being maximum at a certain length D^*. Pathria (1972) and later Pathria and his collaborators (Greenspoon and Pathria 1973, 1974, 1975, Zasada and Pathria 1976, 1977) carried out an extensive, rigorous asymptotic analysis of the thermodynamic behaviour of an ideal Bose gas confined to cuboidal geometries under a variety of boundary conditions. Their calcultions revealed, among several other things, the effects of the boundary conditions employed.

In this paper we are going to investigate the thermodynamics of an ideal non-relativistic Bose gas at very low temperatures confined to the background geometry of an Einstein universe, i.e. the geometry of a three-sphere, S^3. Following the approach of Pathria and his collaborators we study the condensate fraction N_0/N and the specific heat. Two cases are considered, the spin-0 case (scalar field) and the spin-1 case (vector field). The motivation behind these investigations is to extend the problem of Bose–Einstein condensation in finite geometries to curved space. As is well known, the Einstein universe is a closed finite system, described by the structure, $T \otimes S^3$. This feature makes it mathematically tractable, for example the summations involved are only over functions of one integer, as opposed to three for a cuboidal cavity. In § 2 the general formulation of the problem is given. In § 3 the case of spinless gas is considered where we investigate the condensation fraction N_0/N and the specific heat $C_V(N, T, \alpha, a)$. In each case we study the role played by the geometry in smoothing out the singularities of the bulk system. Other features of the system are also discussed. In § 4 the spin-1 gas is considered.

2. Formulation of the problem

In this section we give the general formulae, expressions and definitions needed for the calculation of the total number of particles N, the energy, and the specific heat of an ideal Bose gas. This formulation which was basically designed by Pathria is quite general and is independent of the shape, size, or boundary conditions employed.

The Hamiltonian of an independent Bose–Einstein system is

$$H = \sum_i \epsilon_i a_i^\dagger a_i \tag{1}$$

where $a_i^\dagger$ and a_i are the creation and annihilation operators for the single-particle state which has energy ϵ_i. The following commutation relations hold:

$$[a_i, a_{i'}^\dagger] = \delta_{a',a}; \qquad [a_i, a_{i'}] = 0 = [a_i^\dagger, a_{i'}^\dagger]. \tag{2}$$

The total energy of the system is given by:

$$E = \sum_i d_i \epsilon_i \langle n_i \rangle = \sum_i d_i \epsilon_i (e^{\beta(\epsilon_i - \mu)} - 1)^{-1} \tag{3}$$

where d_i is the degeneracy associated with the ith energy state, $\beta = 1/kT$ and μ is the chemical potential determined by the condition

$$N = \sum_i d_i \langle n_i \rangle = \sum_i d_i (e^{\beta(\epsilon_i - \mu)} - 1)^{-1}. \tag{4}$$

In finite geometries ϵ_i usually depends on the dimensions of the system. This comes about naturally through the influence of the boundary conditions imposed on the wavefunction. Thus one would expect that the thermodynamical functions will be functions of dimensions. The specific heat is defined as

$$C_V = \left(\frac{\partial E}{\partial T}\right)_{N,V}. \tag{5}$$

Now define

$$Z_s = \sum_i d_i \left(\frac{\epsilon_i}{kT}\right)^s \langle n_i \rangle \tag{6}$$

and

$$G_s = -\left(\frac{\partial Z_s}{\partial \alpha}\right)_{T,V} = \sum_i d_i \left(\frac{\epsilon_i}{kT}\right)^s (\langle n_i \rangle + \langle n_i \rangle^2) \tag{7}$$

where $\alpha = -\mu/kT$. If we differentiate equation (3) with respect to T keeping fixed N and V, then using (6) and (7) we get

$$C_V = k[G_2 - (G_1^2/G_0)]. \tag{8}$$

Also it is not difficult (but tedious) to show that,

$$\left(\frac{\partial C_V}{\partial T}\right)_{N,V} = \frac{k}{T}\left\{-2\left(G_2 - \frac{G_1^2}{G_0}\right) + \left[G_3' - 3\left(\frac{G_1}{G_0}\right)G_2' + 3\left(\frac{G_1}{G_0}\right)^2 G_1' - \left(\frac{G_1}{G_0}\right)^3 G_0'\right]\right\} \tag{9}$$

and

$$\begin{aligned}\left(\frac{\partial^2 C_V}{\partial T}\right)_{N,V} = \frac{k}{T^2}\Bigg\{ & 6\left(G_2 - \frac{G_1^2}{G_0}\right) - \frac{3}{G_0}\left[G_2' - 2\left(\frac{G_1}{G_0}\right)G_1' + \left(\frac{G_1}{G_0}\right)^2 G_0'\right]^2 \\ & - 6\left[G_3' - 3\left(\frac{G_1}{G_0}\right)G_2' + 3\left(\frac{G_1}{G_0}\right)^2 G_1' - \left(\frac{G_1}{G_0}\right)^3 G_0'\right] \\ & + \left[G_4'' - 4\left(\frac{G_1}{G_0}\right)G_3'' + 6\left(\frac{G_1}{G_0}\right)^2 G_2'' - 4\left(\frac{G_1}{G_0}\right)^3 G_1'' + \left(\frac{G_1}{G_0}\right)^4 G_0''\right]\Bigg\}\end{aligned} \tag{10}$$

where

$$G_s' = -\left(\frac{\partial G_s}{\partial \alpha}\right)_{T,V} \tag{11}$$

and

$$G_s'' = -\left(\frac{\partial G_s}{\partial \alpha}\right)_{T,V}. \tag{12}$$

The basic problem in these calculations is the evaluation of Z_0, from which it would be possible to deduce all the information needed. This means that we need to perform the summation in (6). In this paper the summation in (6) is not replaced by an integration directly as is the custom when an infinite system is considered. Instead we prefer to perform the sums as they arise to avoid the inaccuracies which accompany the replacement of a sum by an integral. For this purpose we use the following form of the Poisson summation formula (see Titchmarsh 1948)

$$\tfrac{1}{2}f(0) + \sum_{n=1}^{\infty} f(n) = \int_0^{\infty} f(t)\,\mathrm{d}t + 2\sum_{m=1}^{\infty}\int_0^{\infty} f(t)\cos(2m\pi t)\,\mathrm{d}t = I_0 + 2\sum_{m=1}^{\infty} I_m. \tag{13}$$

The first integral on the right-hand side I_0, is the value which one gets if the summation is directly replaced by an integration. Thus this would always represent the bulk result. The second term is the finite size correction and contains the effect of the geometry.

Now that some technical foundations have been clarified we come to define the criteria we use for condensation. In fact there are several definitions in finite systems. These are explained by Goble and Trainor (1966). Pajkowski and Pathria (1977) also considered this question and found that there are three basic types of criteria for condensation in finite systems. These are: (i) macroscopic; (ii) microscopic; (iii) hybrid ones which are based on the ground state properties of the system.

In this paper the condensation region is defined such that a large number of particles is found occupying the ground state. This implies that $\alpha \ll 1$. The condensation temperature is defined according to the macroscopic criterion, i.e., T_0 is the temperature at which C_V is maximum.

Throughout this paper we use units ('absolute units') in which $\hbar = c = G = 1$.

3. Non-relativistic spinless ideal gas

The conformally coupled massive scalar field in the Einstein universe satisfies the equation

$$\Box\phi + \tfrac{1}{6}R\phi + m^2\phi = 0 \tag{14}$$

where $\Box = \nabla_\mu \nabla^\mu$, $R = 6/a^2$ is the scalar curvature and a is the radius of S^3, the spatial part of the Einstein universe. Equation (14) has been considered by many authors (see for example Schrödinger 1938). The eigen-frequencies are found to be

$$\epsilon_n \equiv \omega_n = \frac{1}{a}[(n+1)^2 + m^2a^2]^{1/2} \tag{15}$$

with degeneracy

$$d_n = (n+1)^2 \qquad n = 0, 1, 2, \ldots .$$

The non-relativistic approximation of (15) is

$$\epsilon_n \cong \frac{(n+1)^2}{2ma^2} \tag{16}$$

when the rest mass energy $\epsilon_0 = m$ is omitted.

The number of particles is defined generally in (4). For our specific case we have,

$$N \equiv Z_0 = \sum_{n=1}^{\infty} n^2(e^{\beta' n^2 + \alpha} - 1)^{-1} \tag{17}$$

where

$$\beta' = \frac{1}{2mkTa^2} = \frac{1}{4\pi}\left(\frac{\lambda}{a}\right)^2 \tag{18}$$

with

$$\lambda = \left(\frac{2\pi}{mkT}\right)^{1/2}$$

as the mean thermal wavelength of the particle. It is a measure of the temperature of the system.

To handle the summation in (17) we use the Poisson summation formula (13) by which the 'bulk term' I_0 is evaluated as (see appendix)

$$I_0 = \frac{1}{2}\left(\frac{1}{\beta'}\right)^{3/2} \Gamma(\tfrac{3}{2}) F_{3/2}(\alpha). \tag{19}$$

For the integral I_m, up to terms proportional to $\exp[-(a/\lambda)^2]$ where $a \gg \lambda$, we get

$$I_m = -\frac{\pi}{2}\left(\frac{1}{\beta'}\right)^{3/2} \alpha^{1/2}\, e^{-2my} \tag{20}$$

where

$$y = 2\pi^{3/2}\alpha^{1/2}(a/\lambda). \tag{21}$$

Thus

$$N \equiv Z_0 = \frac{1}{2}\left(\frac{1}{\beta'}\right)^{3/2} (\Gamma(\tfrac{3}{2})F_{3/2}(\alpha) - 2\pi S(2y)) \tag{22}$$

where

$$S(2y) = \sum_{m=1}^{\infty} e^{-2my} = \frac{1}{e^{2y}-1}. \tag{23}$$

From the definition (6) it is possible to show that Z_s can be calculated from the relation (Pathria 1972),

$$\left(\frac{\partial^s Z_s}{\partial \alpha^s}\right)_\beta = \beta^s \left(\frac{\partial^s Z_0}{\partial \beta^s}\right)_\alpha. \tag{24}$$

That is, to get Z_s from Z_0, we have to differentiate Z_0 with respect to β s times (at fixed α) and then integrate with respect to α s times, multiplying the result by β^s at the end. If this is done we get,

$$Z_s = \frac{1}{2}\left(\frac{1}{\beta'}\right)^{3/2} [\Gamma(s+\tfrac{3}{2})F_{3/2}(\alpha) - 2\pi(-1)^s \alpha^{s+\frac{1}{2}} S(2y)]. \tag{25}$$

Having obtained the basic expressions we can proceed to investigate the various thermodynamic functions. If $\alpha \ll 1$ expression (22) yields

$$N \equiv Z_0 = \frac{2}{\pi^{1/2}}\frac{V}{\lambda^3}\left[\tfrac{1}{2}\pi^{1/2}\zeta(\tfrac{3}{2}) - \pi\alpha^{1/2}\left(1 + \frac{2}{e^{2y}-1}\right)\right]$$

or, in terms of the ratio N/V,

$$\frac{N}{V} \equiv \left(\frac{1}{\bar{l}}\right)^3 = \frac{1}{\lambda^3}[\zeta(\tfrac{3}{2}) - 2\pi^{1/2}\alpha^{1/2}\coth y). \tag{26}$$

We have use the expansion

$$F_{3/2}(\alpha) \simeq \zeta(\tfrac{3}{2}) - 2\pi^{1/2}\alpha^{1/2} \tag{27}$$

with $\zeta(s)$ being the Riemann zeta function and $\bar{l}$ is the mean interparticle distance. In the limit $a \to \infty$ we have

$$\left(\frac{N}{V}\right)_{a\to\infty} \equiv \left(\frac{1}{\bar{l}}\right)^3 = \frac{\zeta(\tfrac{3}{2})}{\lambda_0^3(\infty)}. \tag{28}$$

This is just the well known bulk result. Note that we have considered that in the limit $V \to \infty$, N/V remains constant, the usual thermodynamical limit. From (26) we can write,

$$\left(\frac{\lambda}{\bar{l}}\right)^3 = \zeta(\tfrac{3}{2}) - 2\pi^{1/2}\alpha^{1/2}\coth y. \tag{29}$$

Using (28) and the definition of y in (21) we get,

$$\left(\frac{\lambda}{\lambda_0(\infty)}\right)^3 = 1 - \frac{y}{\pi(\zeta(\frac{3}{2}))}\left(\frac{\lambda}{a}\right)\coth y,$$

or

$$\left(\frac{T_0(\infty)}{T}\right)^{3/2} = 1 - \frac{y}{\pi(\zeta(\frac{3}{2}))^{2/3}}\left(\frac{T_0(\infty)}{T}\right)^{1/2}\left(\frac{\bar{l}}{a}\right)\coth y. \tag{30}$$

Set $T/T_0(\infty) \equiv x$, then we have

$$y \coth y = \pi(\zeta(\tfrac{3}{2}))^{2/3}\left(\frac{a}{\bar{l}}\right)x^{1/2}(1 - x^{-3/2}). \tag{31}$$

This equation shows explicitly the dependence of the thermogeometric parameter, y, on temperature (we use the scaled temperature x just to keep the terms involved dimensionless). For a given $a/\bar{l}$ and given x we can solve (31) for y. The set of solutions for $a/\bar{l} = 10$ and $a/\bar{l} = 50$ are given in figure 1 where $(\pi^2 + y^2)^{1/2}$ is plotted

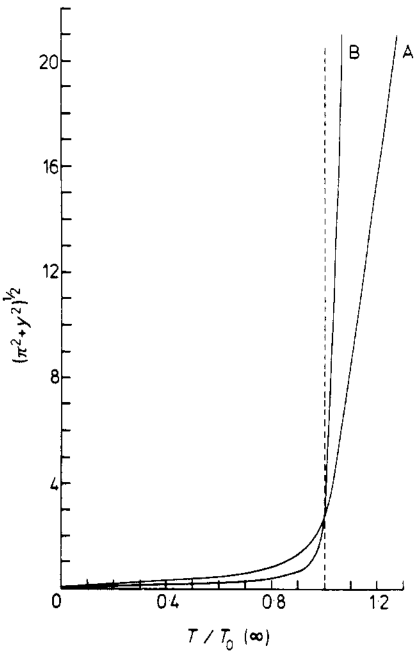

Figure 1. The thermogeometric parameter y for S^3 geometry as a function of the scaled temperature $T/T_0(\infty)$. Curve A is for $N = 1{\cdot}97 \times 10^4$, curve B is for $N = 2{\cdot}46 \times 10^6$. The broken line is for the corresponding bulk behaviour.

against x, and not T, to simplify the figure. Notice that at $T = 0$ we have $y = \pi i$. This result is not surprising, as was pointed out by Pathria, because the zero-temperature limit of the chemical potential, μ, is ϵ_1 which is just $1/2ma^2$ accordingly the limiting value of α is

$$-\frac{1}{2ma^2kT} = -\frac{1}{4\pi}\left(\frac{\lambda}{a}\right)^2$$

and by (21) the corresponding value of y is πi. This means simply that the corresponding value of α is negative. For a given value of $a/\bar{l}$, y becomes real at a certain value of the scaled temperature x. This will be calculated below.

It is clear from figure 1 that as $(a/\bar{l})$ increases y tends to a step function. This makes the thermodynamical functions considered (which are essentially functions of y^2) exhibit a discontinuous behaviour in an infinite system. Herein lies the importance of the thermogeometric parameter y. Now at $x = 1$ ($T = T_0(\infty)$) we have

$$y \coth y = 0. \tag{32}$$

A solution of this equation is

$$y = \mathrm{i}\pi/2.$$

It is important to know at what value of x y^2 beomes zero. Since this will define for us the transition point at which y changes from imaginary to real values. This is known if we solve the equation

$$x^{1/2}[1 - x^{-3/2}] = \frac{1}{\pi}(\zeta(\tfrac{3}{2}))^{-3/2}\left(\frac{\bar{l}}{a}\right). \tag{33}$$

The solution of (34) in the range $x \approx 1$ and for $a/\bar{l} = 10$ is found to be:

$$x = 1{\cdot}011. \tag{34}$$

After the nature of the parameter y has become clear we consider the calculation of the condensate fraction N_0/N. The number of particles in the ground state at temperature T is given by

$$N_0 = [\exp(\beta' + \alpha) - 1]^{-1} = \left[\exp\left(\frac{\pi^2}{y^2} + 1\right)\alpha - 1\right]^{-1}. \tag{35}$$

For $\alpha \ll 1$

$$N_0 = \left[\left(\frac{\pi^2}{y^2} + 1\right)\alpha\right]^{-1}. \tag{36}$$

From (36) and (26) we get,

$$N_0 = N(1 - x^{3/2}) + 2\pi\left(\frac{a}{\lambda}\right)^2 y \coth y + \frac{4\pi^3}{\pi^2 + y^2}\left(\frac{a}{\lambda}\right)^3. \tag{37}$$

The first term in (37) is precisely the expression for N_0 in the bulk system, the second and third terms, therefore, represent the finite size effect, or corrections. Using (31) equation (37) can be further simplified. We get,

$$N_0 = \frac{2\pi}{(\zeta(\frac{3}{2}))^{2/3}} \frac{x}{\pi^2 + y^2}\left(\frac{\bar{l}}{a}\right). \tag{38}$$

This expresses the condensate fraction N_0/N as a function of temperature and dimensions. For a given value $a/\bar{l}$, N_0/N is a function of the scaled temperature x only. This is plotted in figure 2. As the temperature decreases the number of particles in the ground state grows. Here we notice two distinct effects of the finiteness of the system on the condensate fraction, these are: (i) the bulk condensate fraction is enhanced; (ii) the discontinuity at $x = 1$ is smoothed out.

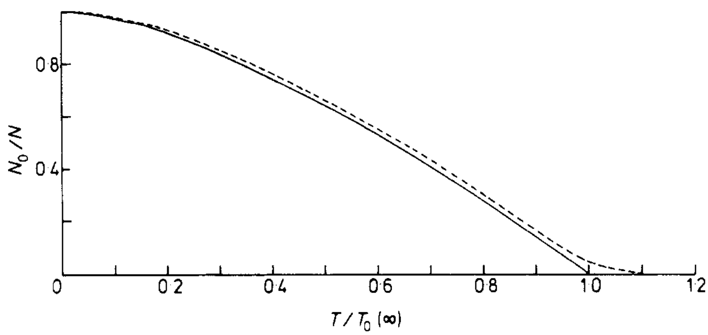

Figure 2. Temperature dependence of the condensate fraction N_0/N for S^3 filled with $1{\cdot}97\times10^4$ spinless particles. The full curve shows the bulk behaviour.

In finite systems the energy levels are discrete. This will shift the spectrum upwards which, in turn, will shrink the occupation number $\langle n_i\rangle$ causing the 'enhancement' recorded in figure 2. The difference between the finite system condensate fraction and the bulk condensate fraction is given in figure 3. This figure shows clearly the smoothing out of the singularity at $x=1$ and shows the fast drop of the condensate fraction after $x=1$. The effect of large radius a is also clear. For large a the difference becomes small, vanishing asymptotically.

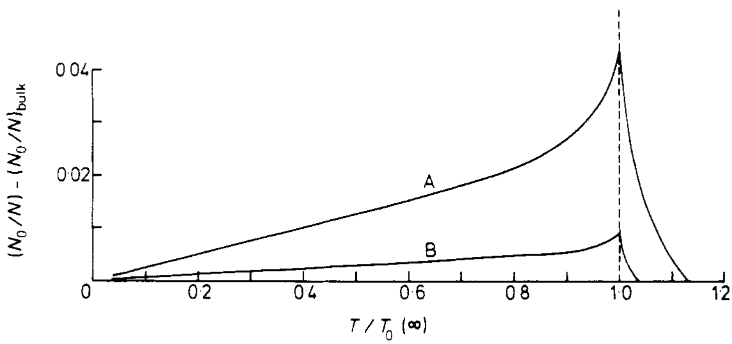

Figure 3. Condensate fraction in excess of the bulk value. Curve A is for $N=1{\cdot}97\times10^4$, curve B is for $N=2{\cdot}46\times10^6$.

3.1. The specific heat

We now wish to investigate the specific heat C_V and discuss the possibility of a transition at the condensation temperature T_0. The general expression for the specific heat is given by equation (8). To calculate C_V we need to know G_0, G_1 and G_2. These functions can be obtained easily from Z_s using relation (5). In the critical region, that is the region in which $\alpha\ll1$ and $y\approx O(1)$), we have, to the first order in $(\bar{l}/a)$,

$$Z_0\simeq\frac{1}{2}\left(\frac{1}{\beta'}\right)^{3/2}(\tfrac{1}{2}\pi^{1/2}\zeta(\tfrac{3}{2})-\pi\alpha^{1/2}\coth y). \tag{39}$$

Therefore

$$G_0\simeq\frac{\pi}{4}\left(\frac{1}{\beta'}\right)^{3/2}(\coth y-y\,\mathrm{cosech}^2\,y)\alpha^{-1/2}. \tag{40}$$

Also from (26) we have

$$Z_1 \simeq \frac{1}{2}\left(\frac{1}{\beta'}\right)^{3/2}[\Gamma(\tfrac{5}{2})(\zeta(\tfrac{5}{2})-\zeta(\tfrac{3}{2})\alpha)+2\pi\alpha^{3/2}S(2y)] \tag{41}$$

which yields

$$G_1 \simeq \frac{1}{2}\left(\frac{1}{\beta'}\right)^{3/2}(\tfrac{3}{4}\pi^{1/2}\zeta(\tfrac{3}{2})-2\pi^{1/2}S(2y)+\tfrac{1}{2}\pi\alpha^{1/2}y\ \mathrm{cosech}^2\ y). \tag{42}$$

Similarly,

$$G_2 \simeq \frac{1}{2}\left(\frac{1}{\beta'}\right)^{3/2}(\tfrac{15}{8}\pi^{1/2}\zeta(\tfrac{5}{2})). \tag{43}$$

Now using (40), (42) and (43) in (8) we obtain

$$\frac{C_V}{Nk} = x^{3/2}\left[\frac{15}{4}\frac{\zeta(\frac{5}{2})}{\zeta(\frac{3}{2})}-\frac{9}{8\pi^2}\frac{(\zeta(\frac{3}{2}))^{4/3}}{x^{1/2}}\frac{y}{\coth y-\mathrm{cosech}^2\ y}\left(\frac{\bar{l}}{a}\right)\right]. \tag{44}$$

This is the expression for the specific heat C_V in the critical region where the geometry of the system will have a distinctive effect. For the infinite space limit ($a\to\infty$) (44) yields

$$\frac{C_0}{Nk} = \frac{15}{4}\frac{\zeta(\frac{5}{2})}{\zeta(\frac{3}{2})}x^{3/2}. \tag{45}$$

At $x = 1$ ($T = T_0(\infty)$) this gives the correct bulk result,

$$\frac{C_0}{Nk} = \frac{15}{4}\frac{\zeta(\frac{5}{2})}{\zeta(\frac{3}{2})}. \tag{46}$$

In an infinite space the specific heat is known to have a cusp-like singularity at $T = T_0(\infty)$. As we can see from (44) this is no longer true in a finite system; the specific heat is a continuous function of the temperature. For this reason one can say that the customary meaning of a sudden condensation is lost in finite systems.

Figure 4 exposes graphically the information contained in equation (44). The specific heat C_V in the critical region, is plotted against the scaled temperature x for a given $\bar{l}/a$ (i.e. for a given number of particles N). In (44) the specific heat depends on y as well as T. We have seen previously that y changes from being imaginary to zero and then becomes real. This has to be taken into account. For imaginary values of y equation (44) becomes

$$\frac{C_V}{Nk} = x^{3/2}\left[\frac{15}{4}\frac{\zeta(\frac{5}{2})}{\zeta(\frac{3}{2})}+\frac{9}{8\pi^2}\frac{y}{x^{1/2}}\frac{(\zeta(\frac{3}{2}))^{4/3}}{\cot y-y\ \mathrm{cosec}^2\ y}\left(\frac{\bar{l}}{a}\right)\right] \tag{47}$$

and for $y = 0$ the second term on the right-hand side vanishes.

In figure 4 we notice that the critical temperature T_0 is shifted to higher values compared with the bulk case. This means that as the radius a of S^3 is increased the critical temperature T_0 is shifted to lower values and it reaches $T_0(\infty)$ as $a\to\infty$. It can also be seen that higher values of a sharpens the specific heat maximum till it becomes a cusp-like singularity in the limit $a\to\infty$.

Now we will show analytically that the specific heat is really a continuous function of temperature and possesses a maximum at $T = T_0$. This is done by equating the first derivative of C_V with respect to the temperature (equation (9)) to zero, thus obtaining

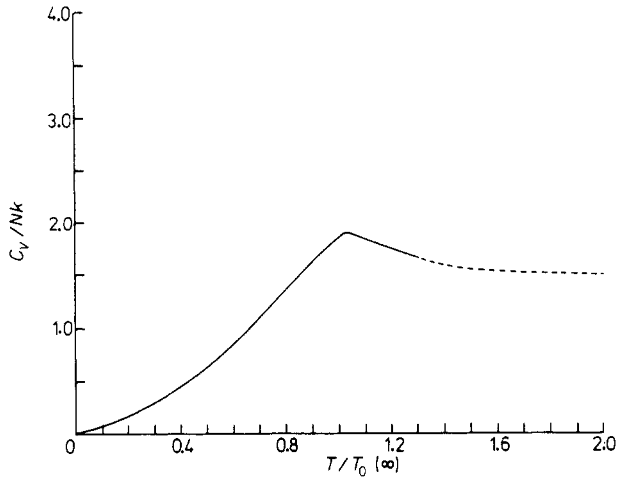

Figure 4. Specific heat against the scaled temperature $T/T_0(\infty)$. This curve shows the continuous behaviour of the specific heat as a function of temperature and the disappearance of the cusp-like singularity of the bulk system.

y_0 which in turn can be used to calculate T_0 using equation (31). The relevant contribution to equation (9) seems to come from G_2, G_1^2/G_0, G_3' and $(G_1/G_0)^2G_0'$ terms only. Expressions for G_2 and G_1^2/G_0 for $\alpha \ll 1$ are given in (42) and (43), we also get

$$\left(\frac{G_1}{G_0}\right)^3 G_0' = \frac{27}{128\pi^{1/2}}\left(\frac{1}{\beta'}\right)^{3/2} (\zeta(\tfrac{3}{2}))^3 \frac{(2\coth y + 2y\,\mathrm{cosech}^2\, y + 4y^2\,\mathrm{cosech}^2\, y\coth y)}{\coth y - y\,\mathrm{cosech}^2\, y}. \tag{48}$$

If (42), (43) and (48) are substituted in (9) we get

$$\frac{T}{Nk}\left(\frac{\partial C_V}{\partial T}\right)_{N,V} \cong \frac{1}{\pi^{1/2}}\left(\frac{\bar{l}}{\lambda}\right)^3\left(\frac{45}{8}\pi^{1/2}\zeta(\tfrac{5}{2}) - \frac{27}{32}\frac{1}{\pi^{1/2}}(\zeta(\tfrac{3}{2}))^3 f(y)\right) \tag{49}$$

where

$$f(y) = \frac{2\coth y + 2y\,\mathrm{cosech}^2\, y + 4y^2\,\mathrm{cosech}^2\, y\coth y}{(\coth y - y\,\mathrm{cosech}^2\, y)^3}. \tag{50}$$

Equating (49) to zero we get ($y = y_0$)

$$\tfrac{45}{8}\pi\zeta(\tfrac{5}{2}) = \tfrac{27}{32}(\zeta(\tfrac{3}{2}))^3 f(y_0),$$

or $f(y_0) = 1{\cdot}577$. The numerical solution of (51) shows that

$$y_0 = 1{\cdot}887. \tag{52}$$

Now we consider the analysis of the critical temperature T_0 and the specific heat C_V in the asymptotic region $a \gg \lambda$. We consider the first-order approximation in $(\bar{l}/a)$ only.

In this region $(\lambda_0(\infty) \simeq \lambda_0(a))$ equation (31) gives,

$$\frac{T_0(a)}{T_0(\infty)} \cong 1 + \frac{2}{3}\frac{y_0\coth y_0}{\pi(\zeta(\frac{3}{2}))^{2/3}}\left(\frac{\bar{l}}{a}\right) = 1 + 0{\cdot}221\left(\frac{\bar{l}}{a}\right). \tag{53}$$

The specific heat maximum $C_0(a)$ can be derived from (46). In the region of interest and to the first order $(\bar{l}/a)$ we have

$$\frac{C_0(a)}{Nk} \cong \frac{15}{4}\frac{\zeta(\frac{5}{2})}{\zeta(\frac{3}{2})} + \frac{15}{4\pi}\frac{\zeta(\frac{5}{2})}{(\zeta(\frac{3}{2}))^{5/3}} y_0 \coth y_0\left(\frac{\bar{l}}{a}\right) - \frac{9}{8\pi^2}\frac{(\zeta(\frac{3}{2}))^{4/3} y_0}{\coth y_0 - y_0 \operatorname{cosech}^2 y_0}\left(\frac{\bar{l}}{a}\right)$$

$$= 1{\cdot}926 - 0{\cdot}134\left(\frac{\bar{l}}{a}\right). \tag{54}$$

The results of (53) and (54) are shown in figures 5 and 6 respectively. We notice that the specific heat maximum is a monotonic function of the size of the system and no maximum is found as is the case for a cuboidal geometry under Dirichlet boundary conditions (Pathria 1972, Greenspoon and Pathria 1973). The specific heat maximum

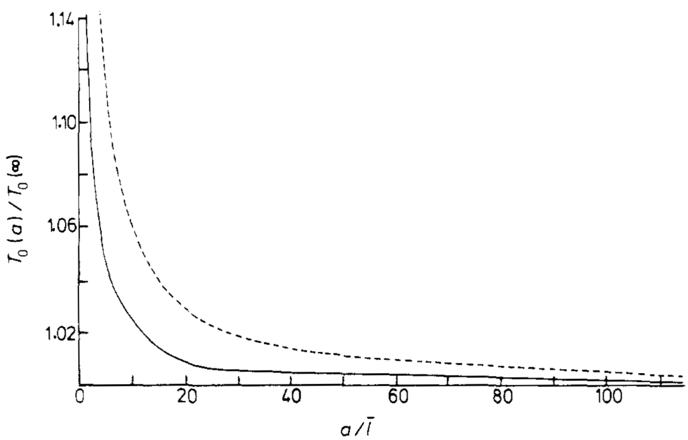

Figure 5. Siee dependence of the condensate temperature $T_0(a)$ at which the specific heat is maximum. The full curve is for spin-0 particles, the broken curve is for spin-1 particles.

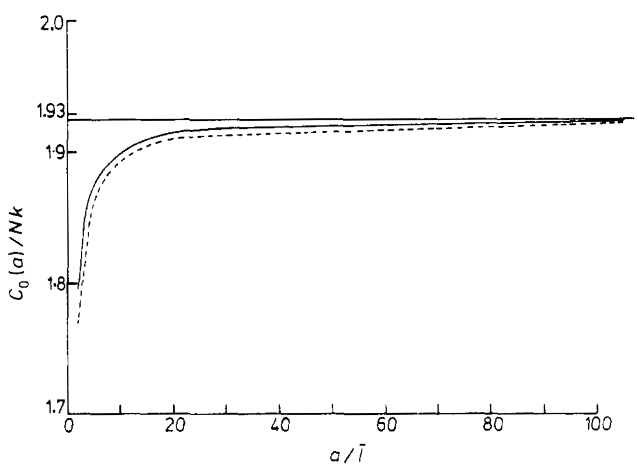

Figure 6. Specific heat maximum $C_0(a)$ as a function of the size of S^3. The full curve is for spin-0 particles, the broken curve is for spin-1 particles. The horizontal line corresponds to the bulk value $C_0(\infty)$.

increases asymptotically to reach the bulk value for large a. The critical temperature drops to lower values as a increases. These results seem to be a general feature of periodic boundary conditions.

4. Non-relativistic spin-1 ideal gas

In this section we consider the Einstein universe to be filled with a non-relativistic ideal gas of spin-1 particles. The equation of motion has been considered by Schrödinger (1938) and the solution yields the following energy spectrum:

$$\epsilon_n = \frac{1}{a}(n^2 + m^2 a^2)^{1/2} \tag{55}$$

with degeneracy

$$d_n = 2(n^2 - 1) \qquad \text{where } n = 2, 3, \ldots . \tag{56}$$

In the non-relativistic limit (with rest mass energy neglected) the energy spectrum (55) gives:

$$\epsilon_n = n^2/2ma^2. \tag{57}$$

Using this spectrum the total number of particles is then given by

$$N = \sum_{n=2}^{\infty} 2(n^2 - 1)(e^{\beta' n^2 + \alpha} - 1)^{-1}. \tag{58}$$

The sum in (58) is performed by using the Poisson summation formula, equation (4), and the result found to be,

$$N = (e^{\alpha} - 1)^{-1} + \left(\frac{1}{\beta'}\right)^{3/2} (\Gamma(\tfrac{3}{2})F_{3/2}(\alpha) - 2\pi\alpha^{1/2} S(2y)) - \left(\frac{1}{\beta'}\right)^{1/2} (\Gamma(\tfrac{1}{2})F_{1/2}(\alpha) + 2\pi\alpha^{-1/2} S(2y)), \tag{59}$$

where symbols are as defined before. The first term arises because the $n = 0$ term in (58) is non-zero. The second term is basically twice that of the scalar case considered in the previous section. The third term arises because of the 'spin curvature' coupling which is described mathematically by the coefficients a_n of the Schwinger–De Witt expansion (De Witt 1965). Both the first and the third terms disappear when $a \to \infty$. In this limit we have

$$\bar{l}^3 = \frac{\lambda_0^3(\infty)}{2\zeta(\frac{3}{2})}. \tag{60}$$

This is just half the known bulk result.

Let us now investigate the information contained in (59). First we look at the thermogeometrical parameter y. We have,

$$\left(\frac{N}{V}\right) = \frac{1}{\bar{l}^3} = \frac{1}{\lambda^3}\left[\frac{2\pi}{y^2}\left(\frac{\lambda}{a}\right) + 2\zeta(\tfrac{3}{2}) - 4\pi^{1/2}\alpha^{1/2}\coth y + \frac{1}{\pi}\left(\frac{\lambda}{a}\right)^2 (\zeta(\tfrac{1}{2}) - \pi^{1/2}\alpha^{-1/2}\coth y)\right] \tag{61}$$

if $a \gg \lambda$) then to first order in $(\bar{l}/a)$ we obtain:

$$2\left(\frac{y}{\pi}+\frac{\pi}{y}\right)\coth y-\frac{2\pi}{y^2}=x^{1/2}(1-x^{-3/2})\left(\frac{\bar{l}}{a}\right)(2\zeta(\tfrac{3}{2}))^{2/3}. \tag{62}$$

This is the equation which expresses the temperature dependence of y. Again, for a given $(\bar{l}/a)$ we can solve for y as a function of the scaled temperature x, where now we have $y = 2\pi \mathrm{i}$ at $x = 0$. This is because in the zero-temperature limit we have

$$\alpha = -\beta'\epsilon_1 = -\frac{2}{ma^2kT} = -\frac{1}{\pi}\left(\frac{\lambda}{a}\right)^2$$

which means that $y(T=0) = 2\pi\mathrm{i}$.

The points x at which $y = 0$ can be found from (62). This gives:

$$2\left(\frac{1}{\pi}+\frac{\pi}{3}\right)=(2\zeta(\tfrac{3}{2}))^{2/3}x^{1/2}(1-x^{-3/2})\left(\frac{a}{\bar{l}}\right). \tag{63}$$

Solving this equation we can find the point x at which y becomes real. This point approaches unity as $a \to \infty$. This significance of the thermogeometrical parameter y is as explained earlier.

4.1. *The condensate fraction N_0/N*

We now wish to see how the number of particles in the ground state ($n = 2$) varies, in proportion to the total number N, as a function of the scaled temperature x.

The number of particles in the ground state at temperature T is given by

$$N_0 = 6[\exp(\beta' n^2 + \alpha)]^{-1}. \tag{64}$$

If $\alpha \ll 1$ this becomes

$$N_0 = \frac{24\pi^3}{4\pi^2 + y^2}\left(\frac{a}{\lambda}\right)^2. \tag{65}$$

From equation (61) it is easy to see that

$$\frac{N_0}{N} = \frac{12\pi}{2\zeta(\tfrac{3}{2}))^{2/3}}\,\frac{x}{4\pi^2+y^2}\left(\frac{\bar{l}}{a}\right). \tag{66}$$

This is an alternative (and simpler) derivation to the one for the corresponding quantity in the spin-0 case. Here again the singularity ($N_0/N = 0$) at $x = 1$ of the bulk system is smoothed out and N_0/N goes to zero asymptotically for large values of x. The bulk condensate fraction is enhanced in this case too.

4.2. *The specific heat*

We can calculate the specific heat in the neighbourhood of the condensate point x_0. The general expression for C_V is given by equation (9). It is clear that we need the functions Z_s. Using (25) and (59) we find

$$Z_s=\left(\frac{1}{\beta'}\right)^{3/2}[\Gamma(s+\tfrac{3}{2})F_{s+\frac{3}{2}}(\alpha)-2\pi(-1)^s\alpha^{s+\frac{1}{2}}S(2y)] \\ -\left(\frac{1}{\beta'}\right)^{1/2}[\Gamma(s+\tfrac{1}{2})F_{s+\frac{1}{2}}(\alpha)+2\pi(-1)^s\alpha^{s-\frac{1}{2}}S(2y)]. \tag{67}$$

This expression is valid for $s>0$. The expression for Z_0 is given by (59). Now we have:

$$G_0\simeq\frac{1}{\alpha^2}+\frac{\pi}{2}\left(\frac{1}{\beta'}\right)^{3/2}\alpha^{-1/2}(\coth y-y\,\mathrm{cosech}^2\,y)-\frac{\pi}{2}\left(\frac{1}{\beta'}\right)^{1/2}\alpha^{-3/2}(\coth y+y\,\mathrm{cosech}^2\,y) \tag{68}$$

$$G_1\simeq\left(\frac{1}{\beta'}\right)^{3/2}\left(\frac{3}{4}\sqrt{\pi}\zeta(\tfrac{3}{2})-\frac{\pi}{2}d^{1/2}(3\coth y-y\,\mathrm{cosech}^2\,y) \\ -\frac{\pi^3}{2y^2}\alpha^{1/2}(\coth y-y\,\mathrm{cosech}^2\,y)\right) \tag{69}$$

and

$$G_2\simeq\frac{15}{8}\left(\frac{1}{\beta'}\right)^{3/2}\sqrt{\pi}\zeta(\tfrac{5}{2}). \tag{70}$$

Using these expressions and the general expression of the specific heat, equation (8), we obtain:

$$\frac{C_V}{Nk}=x^{3/2}\left[\frac{15}{4}\frac{\zeta(\frac{5}{2})}{\zeta(\frac{3}{2})}-\frac{9}{32\pi^2}\frac{y^4}{x^{1/2}}\left(\pi^2+\frac{y^2}{2}(\coth y-y\,\mathrm{cosech}^2\,y) \\ -\frac{\pi^2}{2}y(\coth y+y\,\mathrm{cosech}^2\,y)\right)^{-1}\left(\frac{\bar{l}}{a}\right)\right]. \tag{71}$$

This expression gives the values of C_V in the neighbourhood of the condensation point x_0. A more general expression can be obtained if we consider the exact expressions for Z_s and G_s. For our main purposes expression (71) is sufficient. The specific heat is seen to have a continuous maximum at $x=x_0\simeq 1{\cdot}04$. If we equate the first derivative of C_V with respect to T at constant number N and volume V to zero, we get:

$$\frac{T}{k}\left(\frac{\partial C_V}{\partial T}\right)=\left(\frac{1}{\beta'}\right)^{3/2}\left(\frac{45}{16}\sqrt{\pi}\zeta(\tfrac{5}{2})-\frac{27}{64}\pi\sqrt{\pi}(\zeta(\tfrac{3}{2}))^3f(y)\right) \tag{72}$$

where

$$f(y)=\left(2+\frac{y^3}{4\pi^2}(\coth y+y\,\mathrm{cosech}^2\,y-2y^2\,\mathrm{cosech}^2\,y\coth y)-\frac{y}{4}(3\coth y \\ +3y\,\mathrm{cosech}^2\,y-2y^2\,\mathrm{cosech}^2\,y\coth y)\right)\left[\left(\frac{\pi}{y}\right)^2+\frac{y}{2}(\coth y-y\,\mathrm{cosech}^2\,y) \\ -\frac{\pi^2}{2}(\coth y+y\,\mathrm{cosech}^2\,y)\right]^{-3}. \tag{73}$$

Thus to find the value of y_0 at C_V maximum we have to solve

$$f(y_0)=0{\cdot}119 \tag{74}$$

which yields

$$y_0=0{\cdot}906\mathrm{i}. \tag{75}$$

Again, $|y_0|$ is of order of unity as expected.

We now discuss the asymptotic dependence of the critical temperature $T_0(a)$ on the size of the system. From (62) and in the neighbourhood of $T_0(\infty)$, to the first order in $(\bar{l}/a)$ we have:

$$\frac{T_0(a)}{T_0(\infty)}=1+\frac{4}{3}\frac{1}{(2\zeta(\frac{3}{2}))^{2/3}}\left[\left(\frac{y_0}{\pi}+\frac{\pi}{y_0}\right)\coth y_0-\frac{\pi}{y_0^2}\right]\left(\frac{\bar{l}}{a}\right) \tag{76}$$

and for the value of y_0 in (75) we get

$$\frac{T_0(a)}{T_0(\infty)}=1+0{\cdot}592\left(\frac{\bar{l}}{a}\right). \tag{77}$$

The results are plotted in figure 5.

It is also interesting to know how the specific heat maximum (or rather (C_0/Nk)) changes with the size of the system. From (21) and for imaginary values of y we have:

$$\frac{C_0(a)}{Nk}=x^{3/2}$$
$$\times\left(1{\cdot}926-\frac{9}{32\pi^2}\frac{y_0^4(\bar{l}/a)}{\pi^2-\frac{1}{2}y_0^3(\cot y_0-y_0\operatorname{cosec}^2 y_0)-\frac{1}{2}\pi^2 y_0(\cot y_0+y_0\operatorname{cosec}^2 y_0)}\right). \tag{78}$$

For the given value of y_0 and to first order in $(\bar{l}/a)$ we obtain

$$\frac{C_0(a)}{Nk}\cong{\cdot}926-0{\cdot}308\left(\frac{\bar{l}}{a}\right). \tag{79}$$

Therefore the specific heat maximum increases with the size of the system ultimately reaching the bulk value for very large radius a. The results are shown in figure 6.

5. Discussion

In the previous sections we have studied the behaviour of an ideal non-relativistic Bose gas confined to the background geometry of an Einstein universe. The general outcome of the calculation was in fact within what one would expect of the finite geometry effects, and in this respect we find that the results are consistent with the scaling theory for finite size effects (Fisher and Barber 1972).

The scaling theory works if and only if

$$a/\bar{l}\gg 1$$

and

$$\frac{T-T_0(\infty)}{T_0(\infty)}\ll 1 \tag{80}$$

Thus, to the first order in $(\bar{l}/a)$ one would expect the specific heat $C_V(T)$, in the critical region, to be governed by the variable

$$z = t\left(\frac{a}{\bar{l}}\right) \tag{81}$$

where

$$t = \frac{T - T_0(\infty)}{T_0(\infty)}. \tag{82}$$

This is indeed true. From equation (31) we can write,

$$z = t\left(\frac{a}{\bar{l}}\right) \cong \frac{2}{3} - \frac{1}{\pi(\zeta(\frac{3}{2}))^{2/3}}(F(y) - F(y_0)) \cong \frac{2}{3}\frac{1}{\pi(\zeta(\frac{3}{2}))^{2/3}}(y \coth y - y_0 \coth y_0). \tag{83}$$

Clearly, the scaling variable z is a function of y alone (e.g. for $y = y_0$, $z = 0$). It follows that y is in turn a function of z alone. This means that the specific heat $C_V(T)$ given by (46) is governed by the variable z only.

Similar arguments apply in the case of spin-1 discussed in § 4.

The curved nature and compactness of the Einstein space is reflected in the energy spectrum which in particular, shows that the ground state energy is non-zero. This is caused by the conformal invariance factor $R/6$, in the scalar wave equation, which shifts the energy spectrum upwards. A non-zero ground state energy makes the calculations run in an analogous fashion to that of a box with Dirichlet boundary conditions ($\psi(L) = 0$), the thermogeometrical parameter y is imaginary for $T < T_0(\infty)$, but the asymptotic analysis of the specific heat shows (see figure 6) that the specific heat maximum is a monotonic function of the size of the system with none of the special features (e.g. a maximum at a certain size) of the box under Dirichlet boundary conditions. This is because basically we are using periodic boundary conditions and not the Dirichlet ones.

It is observed that the specific heat maximum occurs when y acquires a certain characteristic value which is independent of the size of the system. This supports the law of correspondence first noticed by Greenspoon and Pathria (1974).

A look at the results of the asymptotic analysis for the spin-0 and spin-1 cases shows that the differences are only minor quantitative ones and not qualitative. This is attributed to the differences in the ground state energies and degeneracies.

An extension of the present problem into the relativistic regime may be interesting since it is observed (Landsberg and Dunning-Davies 1965) that there are some fundamental departures from the non-relativistic case. This may be reported elsewhere.

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Appendix. Calculations of Z_0

We have

$$N = Z_0 = \sum_{n=1}^{\infty} n^2 (e^{\beta' n^2 + \alpha} - 1)^{-1}. \tag{A.1}$$

Using the Poisson summation formula, (13), the sum in (A.1) can be written,

$$Z_0 = \int_0^\infty \frac{t^2\, dt}{e^{\beta' t^2 + \alpha} - 1} + 2 \sum_{m-1}^{\infty} \int_0^\infty \frac{t^2 \cos(2m\pi t)}{e^{\beta' t^2 + \alpha} - 1}\, dt = I_0 + 2 \sum_{m=1}^{\infty} I_m, \tag{A.2}$$

where

$$I_0 = \tfrac{1}{2}(1/\beta')^{3/2} \Gamma(\tfrac{3}{2}) F_{3/2}(\alpha) \tag{A.3}$$

with $F_\sigma(\alpha)$ being the Bose–Einstein function discussed by Robinson (1951). The Dirichlet series of $F_\sigma(\alpha)$ is given by

$$F_\sigma(\alpha) = \sum_{n=1}^{\infty} n^{-\sigma} e^{-n\alpha}. \tag{A.4}$$

For $\alpha \ll 1$ we use the following expansions:

$$F_{1/2}(\alpha) = 1{\cdot}773\alpha^{1/2} - 1{\cdot}460 = -\zeta(\tfrac{1}{2}) + \sqrt{\pi}\alpha^{-1/2}$$

$$F_{3/2}(\alpha) = 2{\cdot}612 - 3{\cdot}545\alpha^{1/2} = \zeta(\tfrac{3}{2}) - 2\pi^{1/2}\alpha^{1/2}$$

$$F_{5/2}(\alpha) = 1{\cdot}342 - 2{\cdot}612\alpha = \zeta(\tfrac{5}{2}) - \zeta(\tfrac{3}{2})\alpha.$$

Now concerning the I_m integral we have,

$$I_m(\alpha) = \int_0^\infty \frac{t^2 \cos(2m\pi t)}{e^{\beta' t^2 + \alpha} - 1}\, dt$$

$$= -\frac{\partial}{\partial \beta'} \sum_{l=1}^{\infty} \frac{e^{-l\alpha}}{l} \int_0^\infty e^{-l\beta' t^2} \cos(2m\pi t)\, dt$$

$$= \frac{\pi^{1/2}}{2} \left(\frac{1}{\beta'}\right)^{3/2} \sum_{l=1}^{\infty} \frac{e^{-l\alpha}}{l^{3/2}} \left(\frac{1}{2} - \frac{\pi^2 m^2}{l\beta'}\right) e^{-m^2\pi^2/l\beta'}$$

where we have used (Gradshteyn and Ryzhik (1965)

$$\int_0^\infty \cos(bx)\, e^{-\beta x^2}\, dx = \frac{1}{2}\left(\frac{\pi}{\beta}\right)^{1/2} e^{-b^2/4\beta}$$

and define $y = 2\pi^{3/2}\alpha^{1/2}(a/\lambda)$ and $p = l\alpha$, then

$$\frac{m^2\pi^2}{l\beta'} = \frac{1}{p} m^2 y^2,$$

so that,

$$2 \sum_{m=1}^{\infty} I_m(\alpha) = \pi^{1/2} \left(\frac{1}{\beta'}\right)^{3/2} \sum_{l=1}^{\infty} \sum_{m=1}^{\infty} \frac{e^{-p}}{l^{3/2}} \left(\frac{1}{2} - \frac{m^2 y^2}{p}\right) e^{-m^2 y^2/p}.$$

Here we impose some approximations. Change the summation over l into an integration over p to give,

$$2\sum_{m=1}^{\infty} I_m(\alpha)=\pi^{1/2}\alpha^{1/2}\left(\frac{1}{\beta'}\right)^{3/2}\sum_{m=1}^{\infty}\int_0^{\infty}\frac{e^{-p}}{p^{3/2}}\left(\frac{1}{2}-\frac{m^2y^2}{p}\right)e^{-m^2y^2/p}\,dp.$$

Using the Laplace transform (Abramowitz and Stegun 1968)

$$\mathscr{L}\left(\frac{1}{t^{3/2}}e^{-k^2/4t}\right)=\frac{2\pi^{1/2}}{k}e^{-k\sqrt{s}}$$

we get

$$2\sum_{m=1}^{\infty} I_m(\alpha)=-\pi\left(\frac{1}{\beta'}\right)^{3/2}\alpha^{1/2}\sum_{n=1}^{\infty}e^{-2my} \tag{A.5}$$

so that

$$Z_0=\frac{1}{2}\left(\frac{1}{\beta'}\right)^{3/2}(\Gamma(\tfrac{3}{2})F_{3/2}(\alpha)-\pi\alpha^{1/2}S(2y)) \tag{A.6}$$

where

$$S(2y)=\sum_{m=1}^{\infty}e^{-2my}=\frac{1}{e^{2y}-1} \tag{A.7}$$

Note that in converting the summation over l into an integration over p we have implied actually an approximation of the order of $\exp[-(a/\lambda)^2]$ which is negligibly small if $a \gg \lambda$.

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